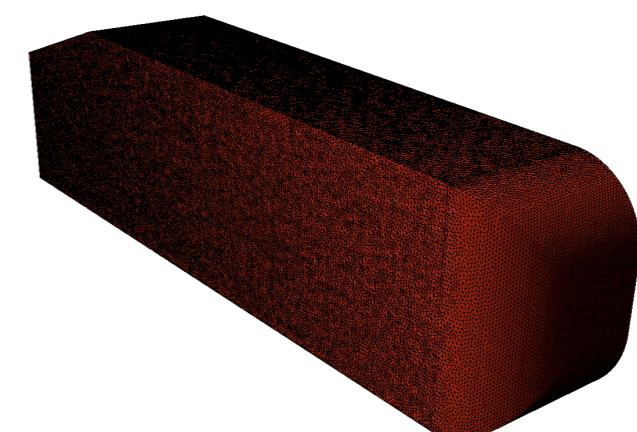
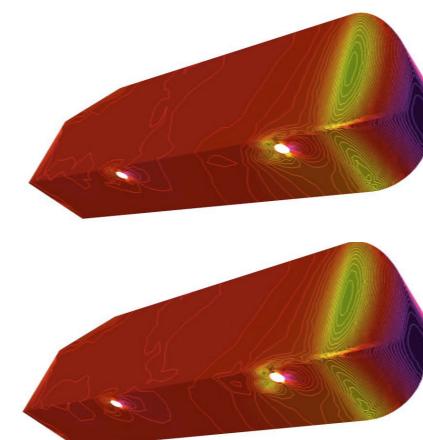
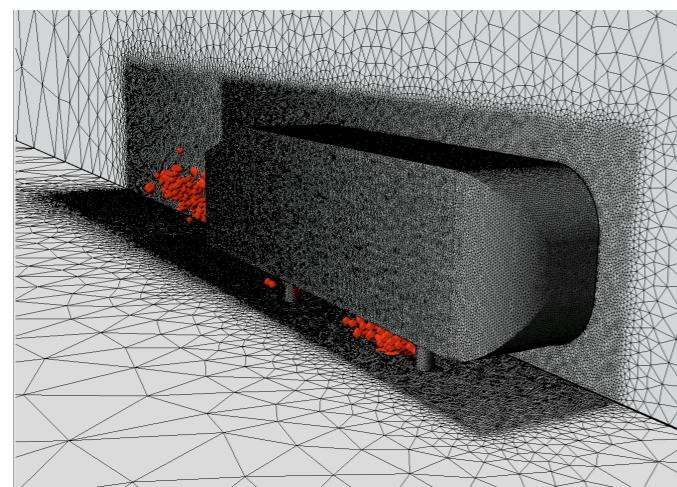




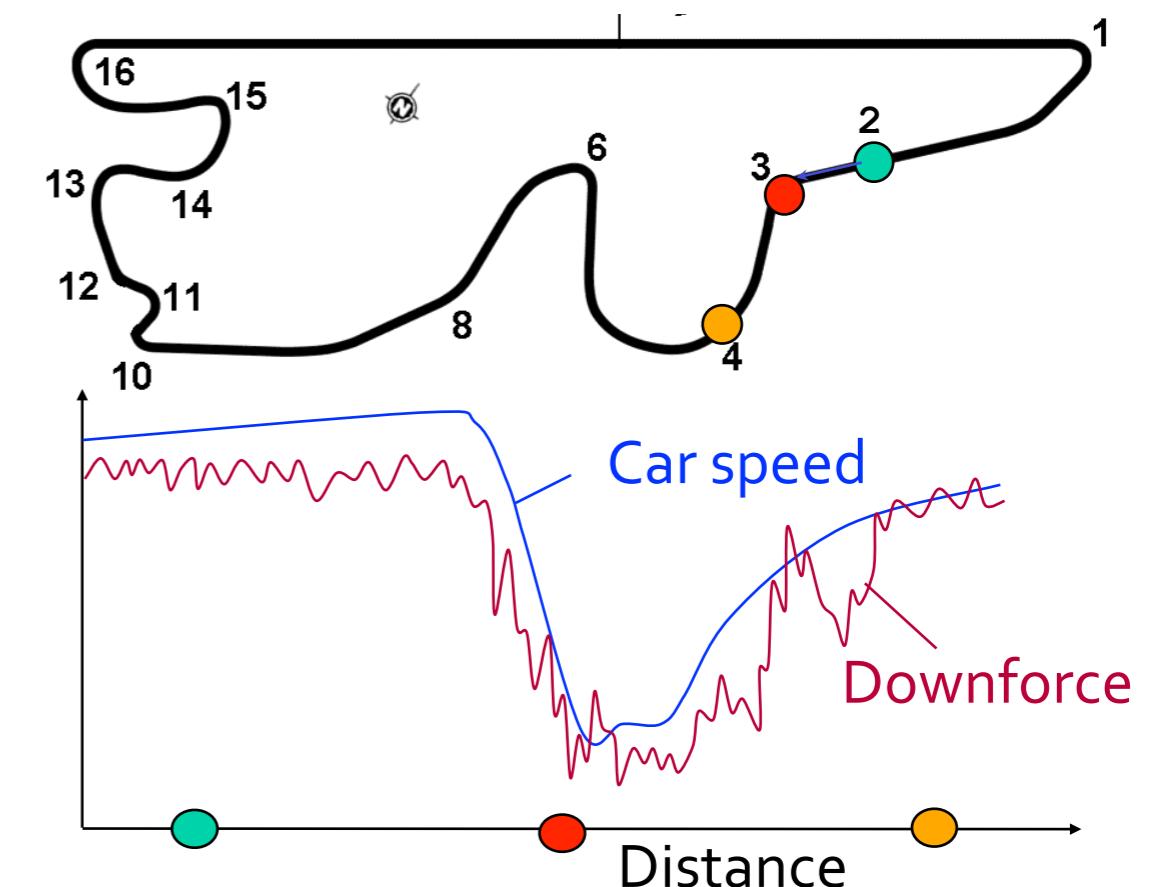
The GNAT nonlinear model reduction method and its application to fluid dynamics problems



Kevin Carlberg, Julien Cortial, David Amsallem, Matt Zahr,
& Charbel Farhat

6th AIAA Theoretical Fluid Mechanics Conference

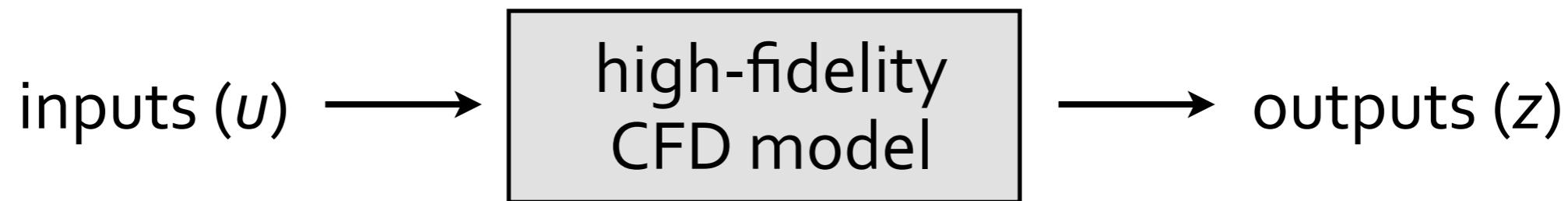
June 27, 2011



- *Goal:* many rapid analyses *during* season (online)
- *Opportunity:* costly analyses *before* season (offline)

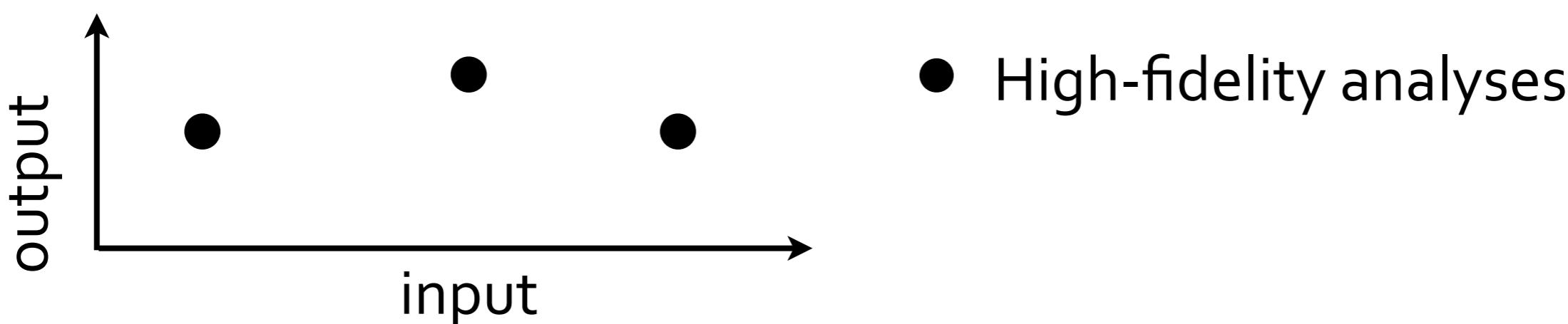


Offline/online ROM strategy



1) Offline

- Costly “training” analyses with high-fidelity model
- Collect snapshots
- Build reduced-order model





Offline/online ROM strategy

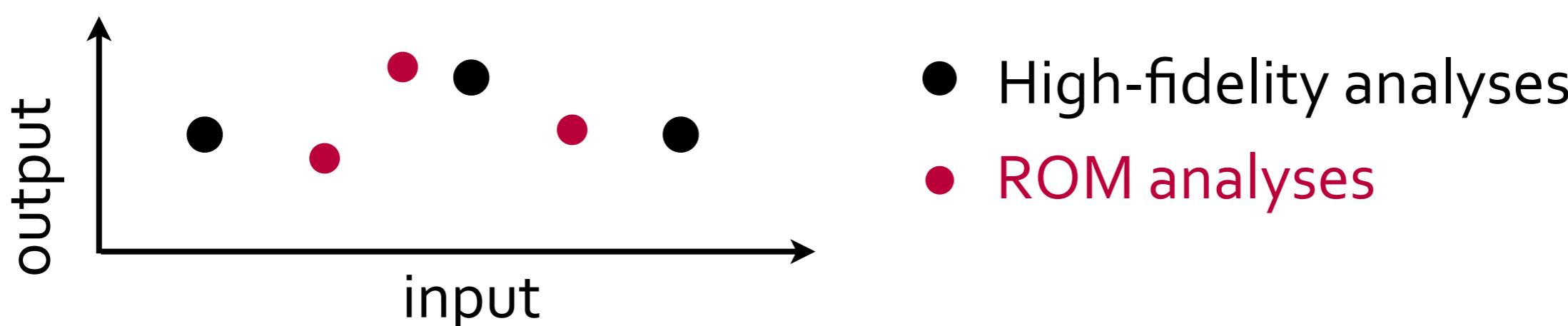


1) Offline

- Costly “training” analyses with high-fidelity model
- Collect snapshots
- Build reduced-order model

2) Online

- Rapid analyses with **reduced-order model (ROM)**



Outline

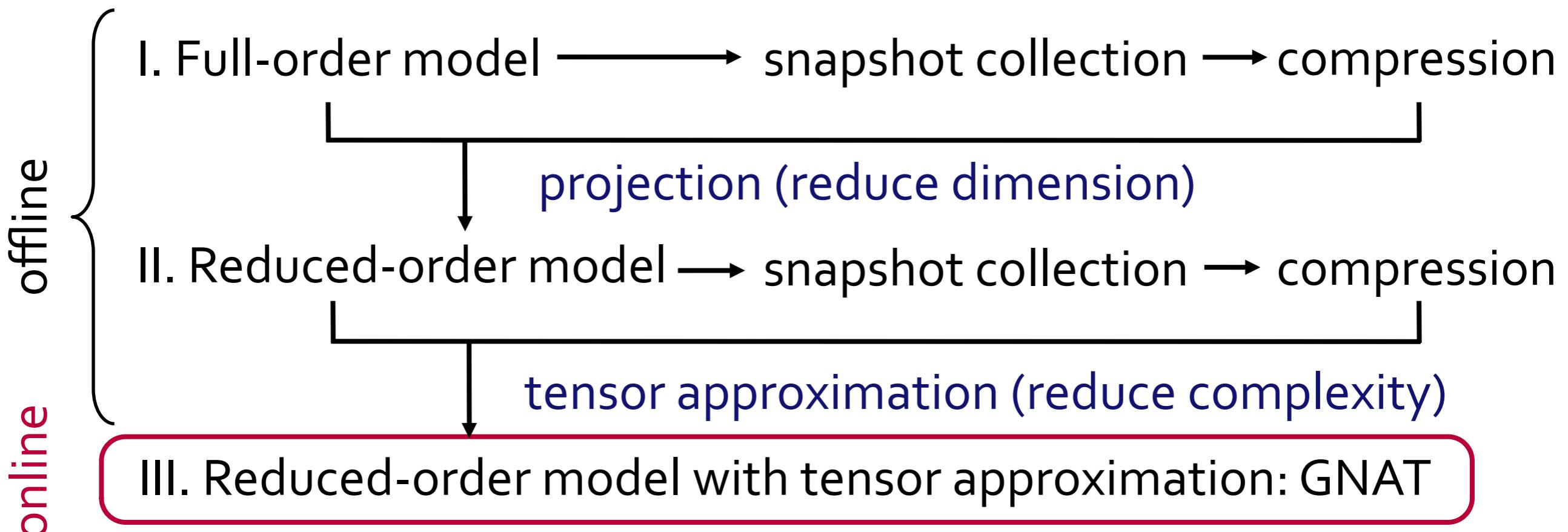
- Introduction
- Gauss–Newton with Approximated Tensors (GNAT)
- Sample mesh
- Examples

- Introduction
- Gauss–Newton with Approximated Tensors (GNAT) [Carlberg *et al.*, 2011]
 - Approach
 - Projection
 - Tensor approximation
- Sample mesh
- Examples

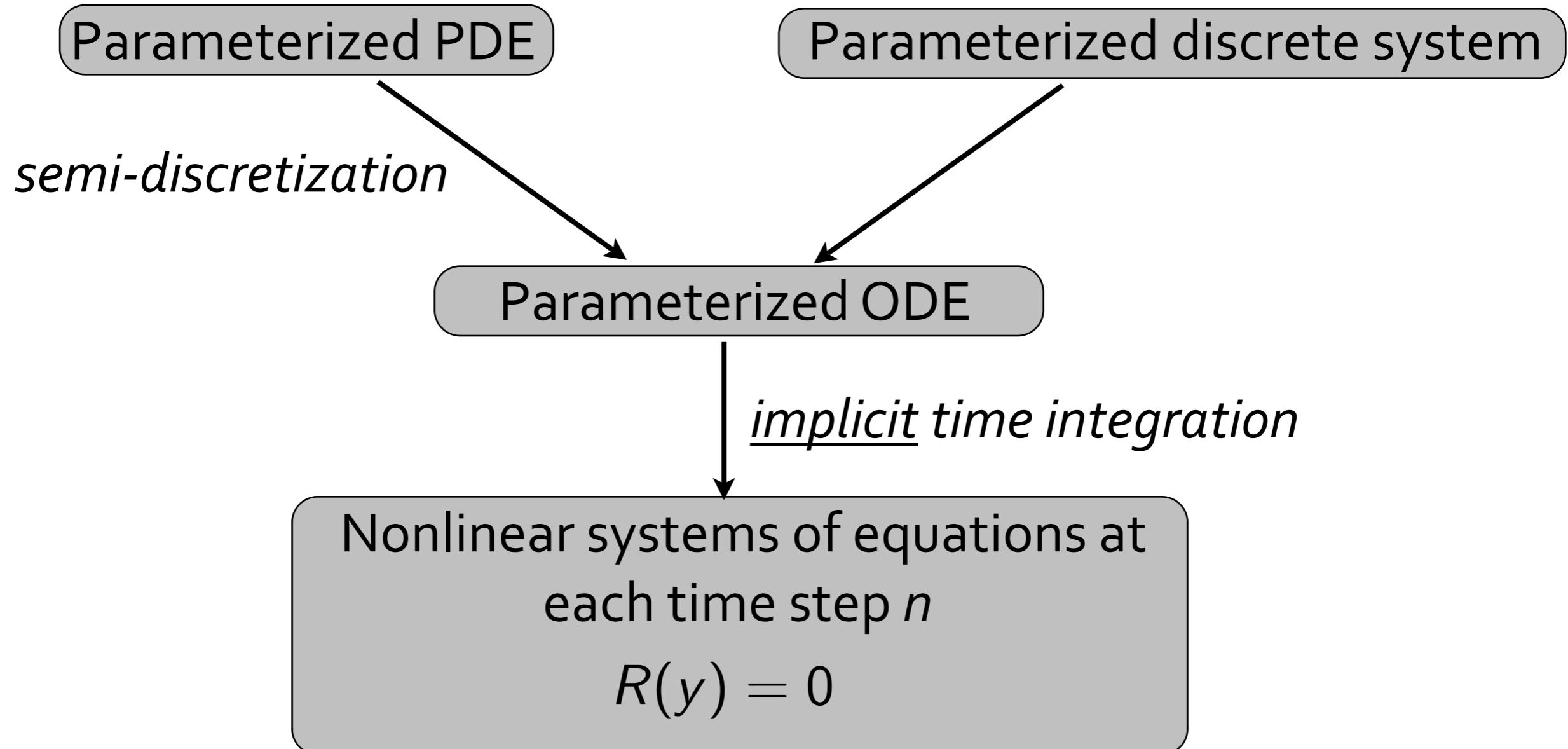




Model hierarchy



1. **Consistency**: in the limit of *no compression*, exactly reproduce the previous model's solution at training inputs
2. **Optimality**: minimize error over approximation space
 - + *A priori* convergence: error decreases monotonically as space expands



- + Framework is **general**, enables **optimal projection** (per time step)
- Less convenient: ROM not an ODE

Outline

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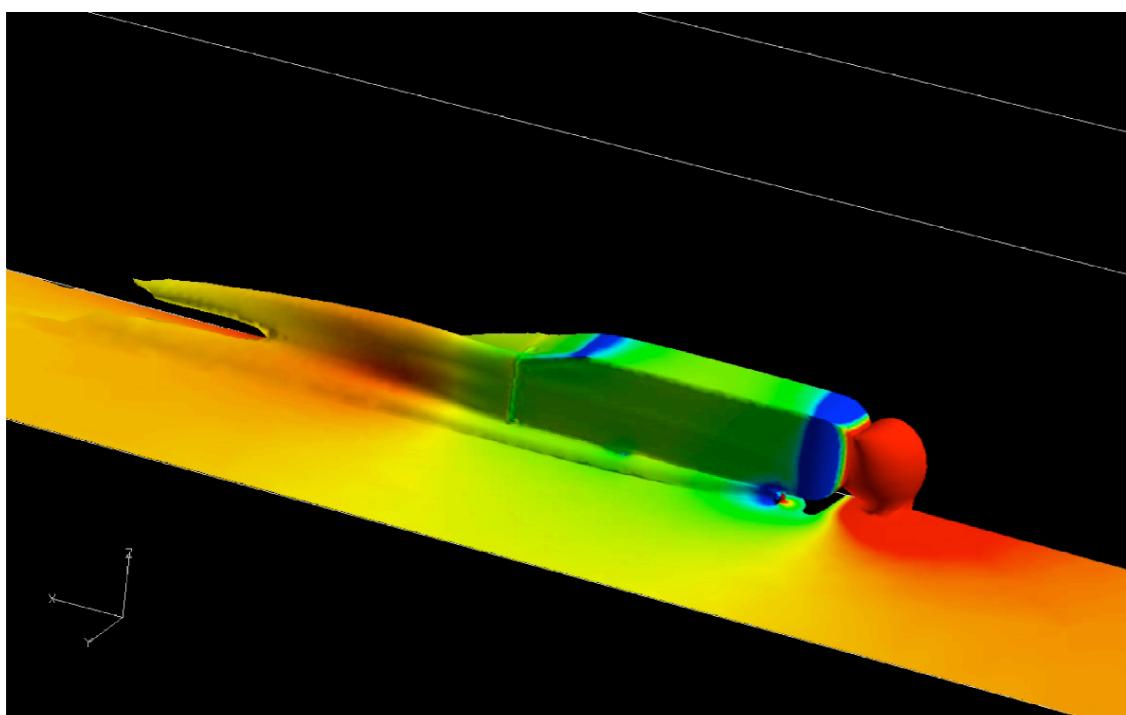


Projection



Offline

1. Collect **consistent** state snapshots

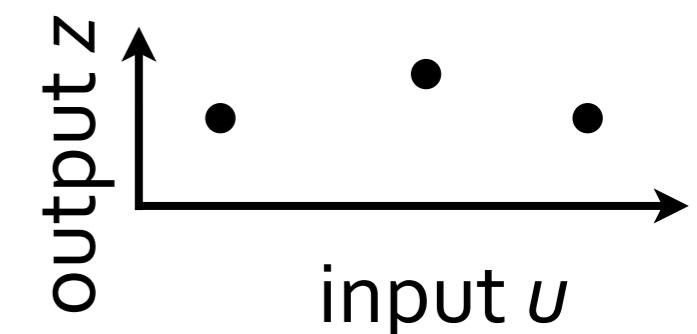


2. Compression

- a. compute singular value decomposition

- b. truncate to obtain POD basis

$$U$$



$$Y =$$

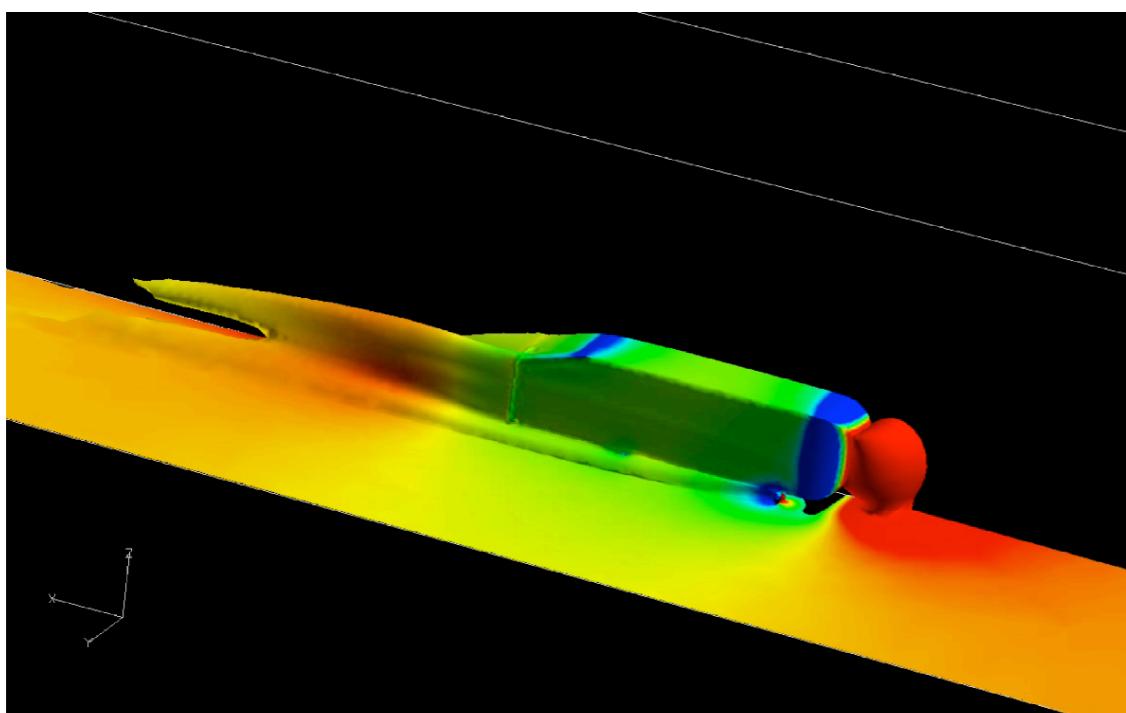
$$Y = U \Sigma V^T$$



Projection

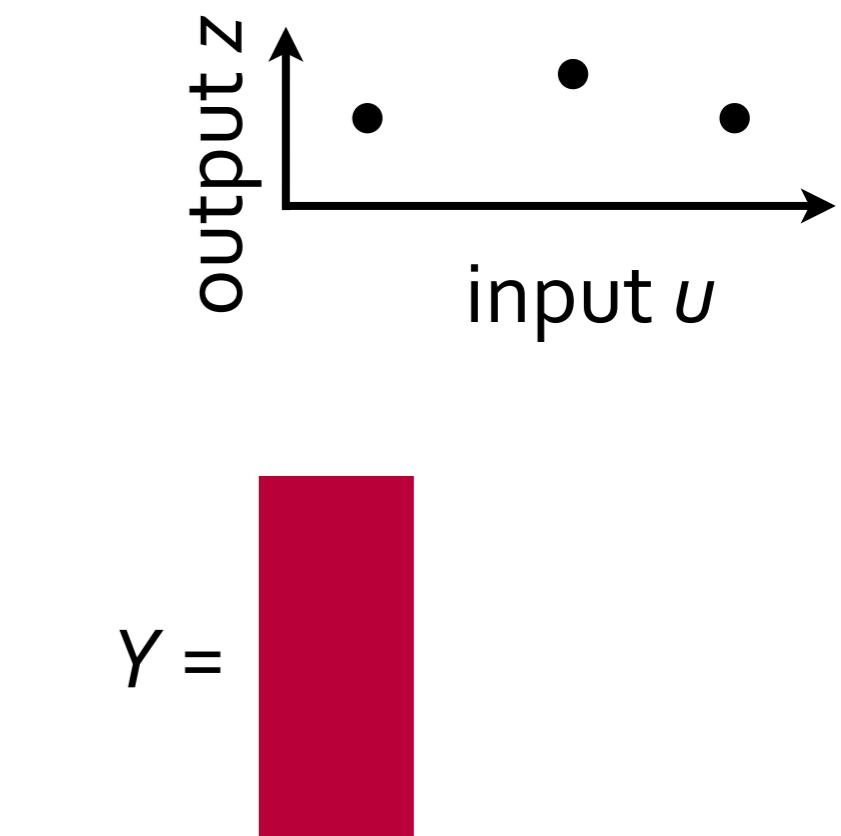
Offline

1. Collect **consistent** state snapshots



2. Compression

- a. compute singular value decomposition



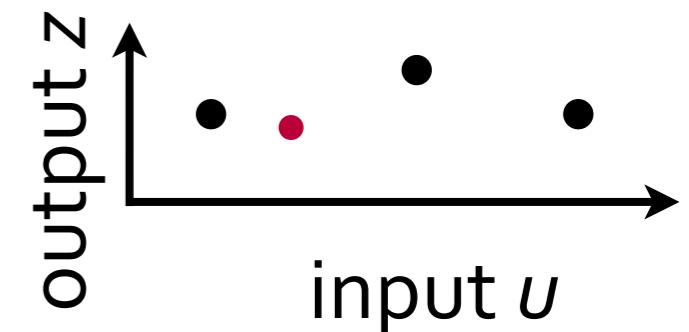
$$Y =$$

$$Y = U \Sigma V^T$$

- b. truncate to obtain POD basis

$$\Phi_y$$

$$R(y) = 0$$



reduce # unknowns

$$y \approx y_0 + \Phi_y y_r$$

$$\boxed{\quad} \approx \boxed{\quad} + \boxed{\quad}$$

$$y = \arg \min_{x \in y_0 + \text{range}(\Phi_y)} \|R(x)\|_2$$

• solve by Gauss–Newton method

- + Least-squares optimal at each time step
- Requires fully discrete framework
- Similar ideas: Bui-Thanh *et al.*, 2009; Legresley and Alonso, 2001; Constantine *et al.*, 2011

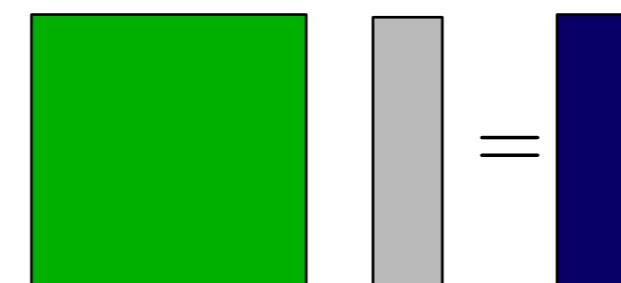


Bottleneck



- Tier II iterations are

$$[J^{(k)}\Phi_y]^T [J^{(k)}\Phi_y] \rho^{(k)} = -[J^{(k)}\Phi_y]^T R^{(k)}$$



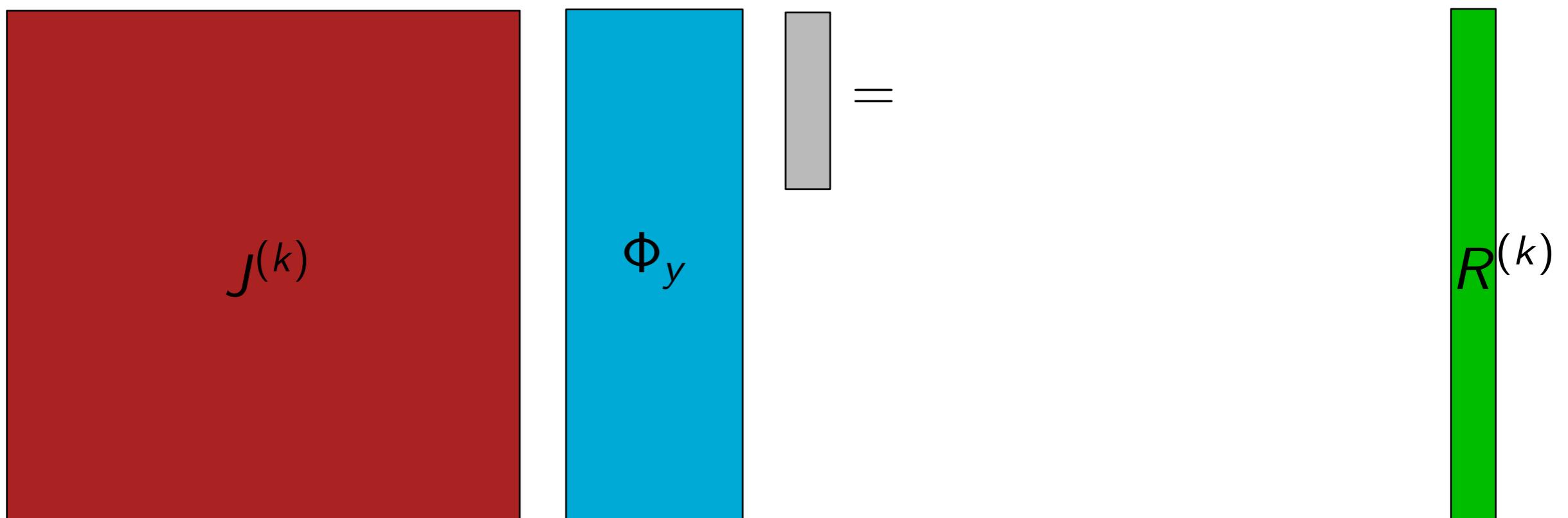


Bottleneck



- Tier II iterations are

$$[J^{(k)}\Phi_y]^T [J^{(k)}\Phi_y]\rho^{(k)} = -[J^{(k)}\Phi_y]^T R^{(k)}$$





Bottleneck



- Tier II iterations are

$$[J^{(k)}\Phi_y]^T [J^{(k)}\Phi_y] \rho^{(k)} = -[J^{(k)}\Phi_y]^T R^{(k)}$$

$$\begin{matrix} \text{Yellow Box} & = & \text{Green Box} \\ J^{(k)}\Phi_y & & R^{(k)} \end{matrix}$$

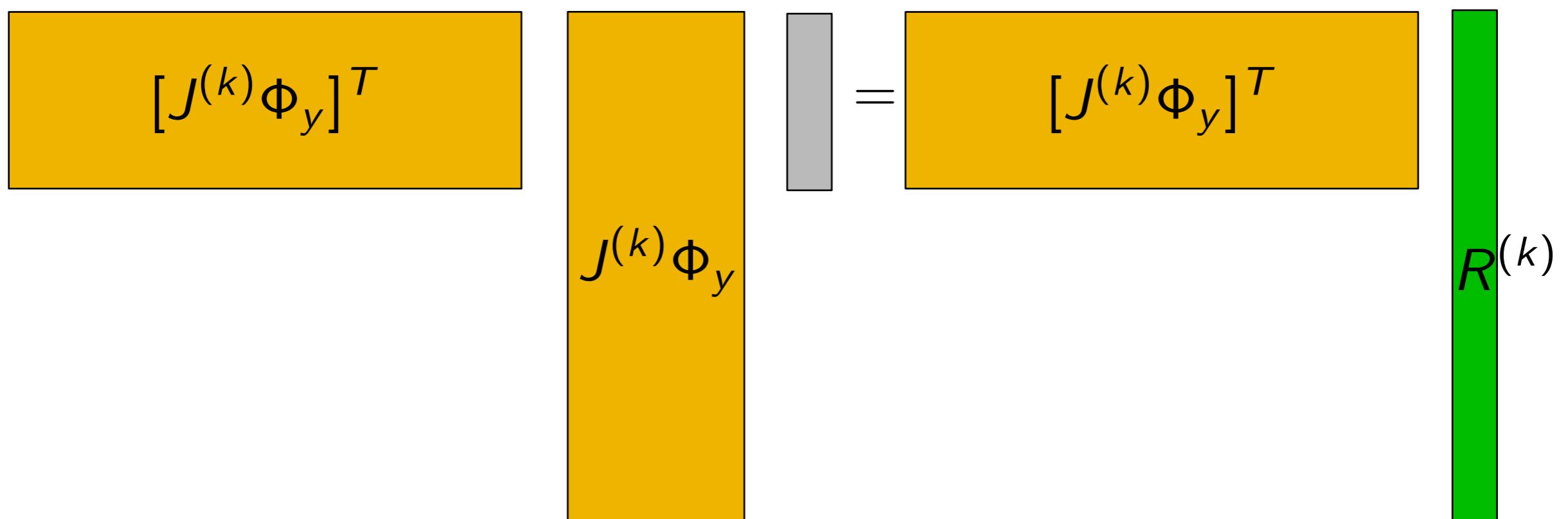


Bottleneck



- Tier II iterations are

$$[J^{(k)}\Phi_y]^T [J^{(k)}\Phi_y] \rho^{(k)} = -[J^{(k)}\Phi_y]^T R^{(k)}$$



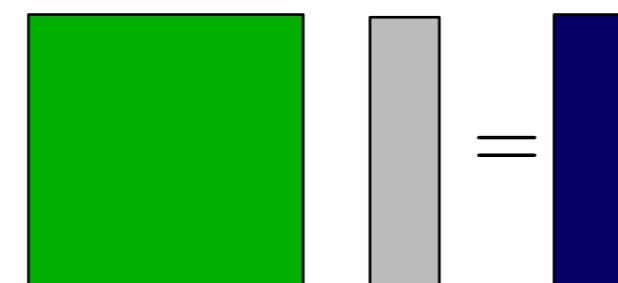


Bottleneck



- Tier II iterations are

$$[J^{(k)}\Phi_y]^T [J^{(k)}\Phi_y] \rho^{(k)} = -[J^{(k)}\Phi_y]^T R^{(k)}$$



- Operations scale with N (expensive)



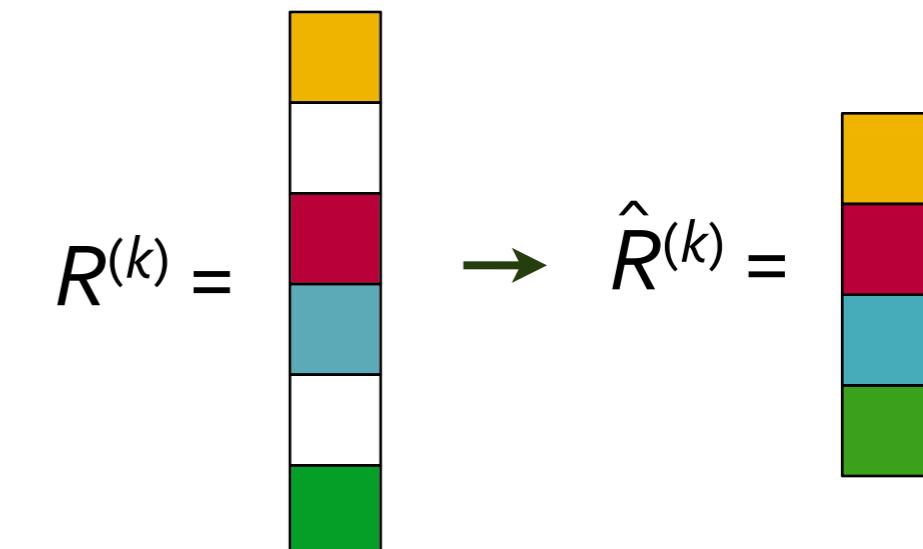
Outline



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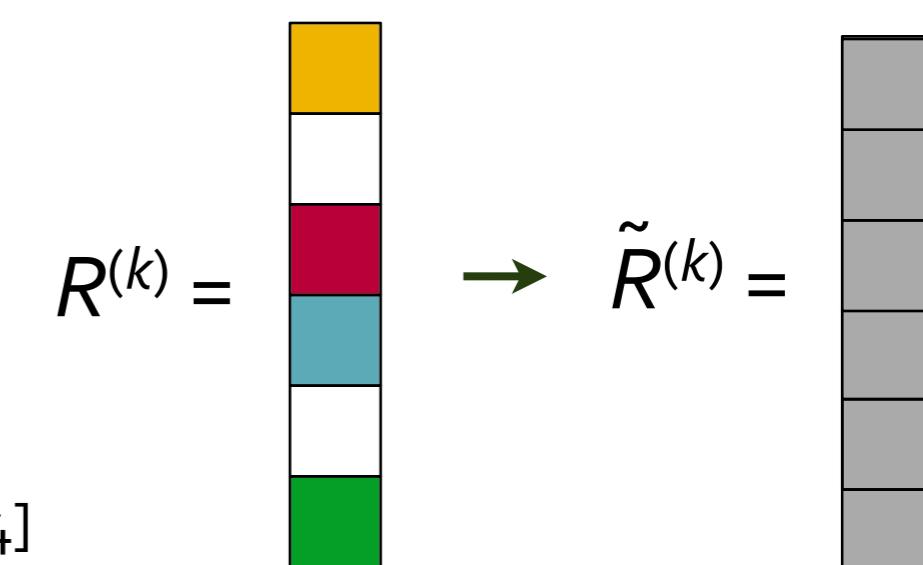
1. Collocation

- omits many elements
- Galerkin projection [Astrid, 2004]
- least squares [LeGresley and Alonso, 2001]

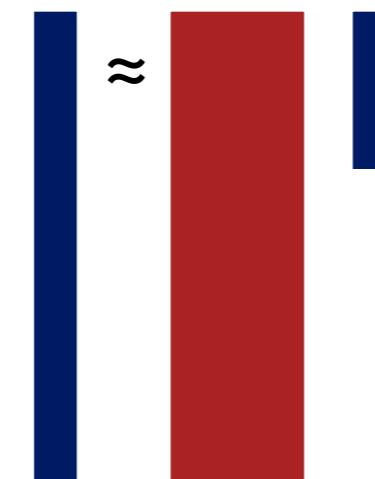


2. Function reconstruction

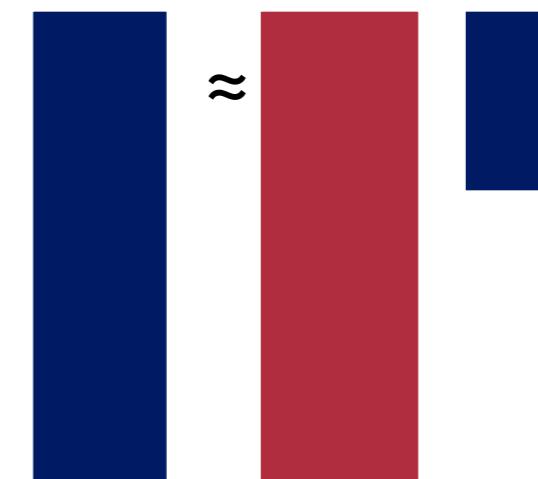
- + approximates entire vector
- $R^{(k)} \in \text{range}(\Phi_y)$ [Astrid *et al.*, 2008; Bos *et al.*, 2004]
- “discrete empirical interpolation” [Chaturantabut *et al.*, 2010; Galbally *et al.*, 2010]
- GNAT [Carlberg *et al.*, 2010]
 - + only consistent method



$$R^{(k)} \approx \Phi_R R_r^{(k)}$$



$$J^{(k)} \Phi_y \approx \Phi_J J_r^{(k)}$$



- Consistency
 - Φ_R, Φ_J computed offline by POD with specific snapshots

- Optimality
 - $R_r^{(k)}, J_r^{(k)}$ computed online by “Gappy POD”
 - requires computing only **a few rows** of residual/Jacobian

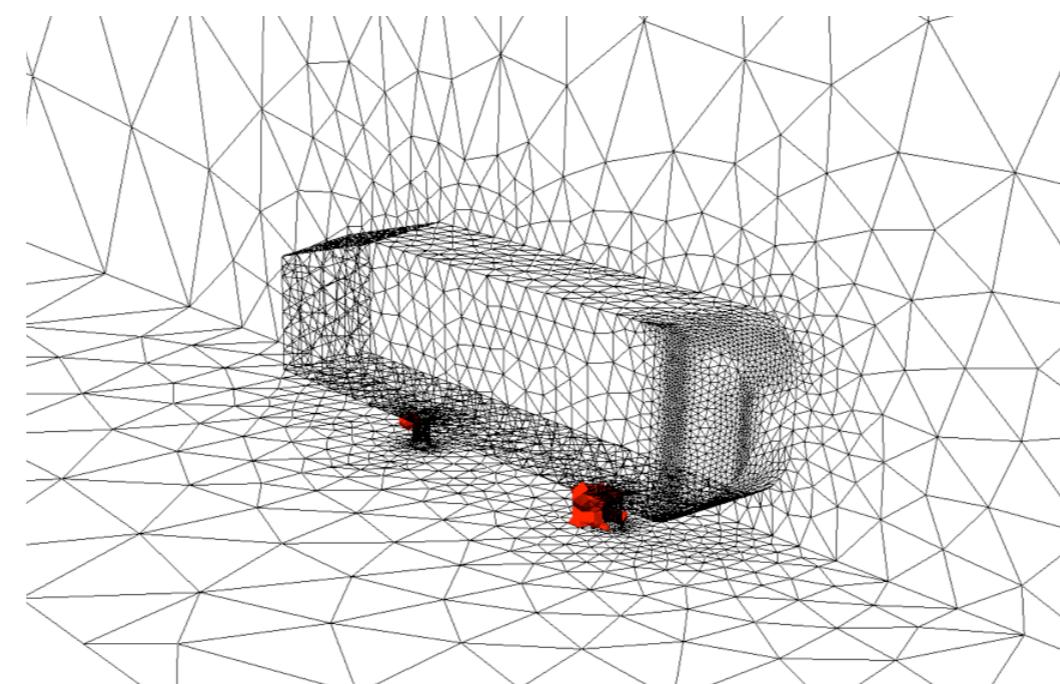


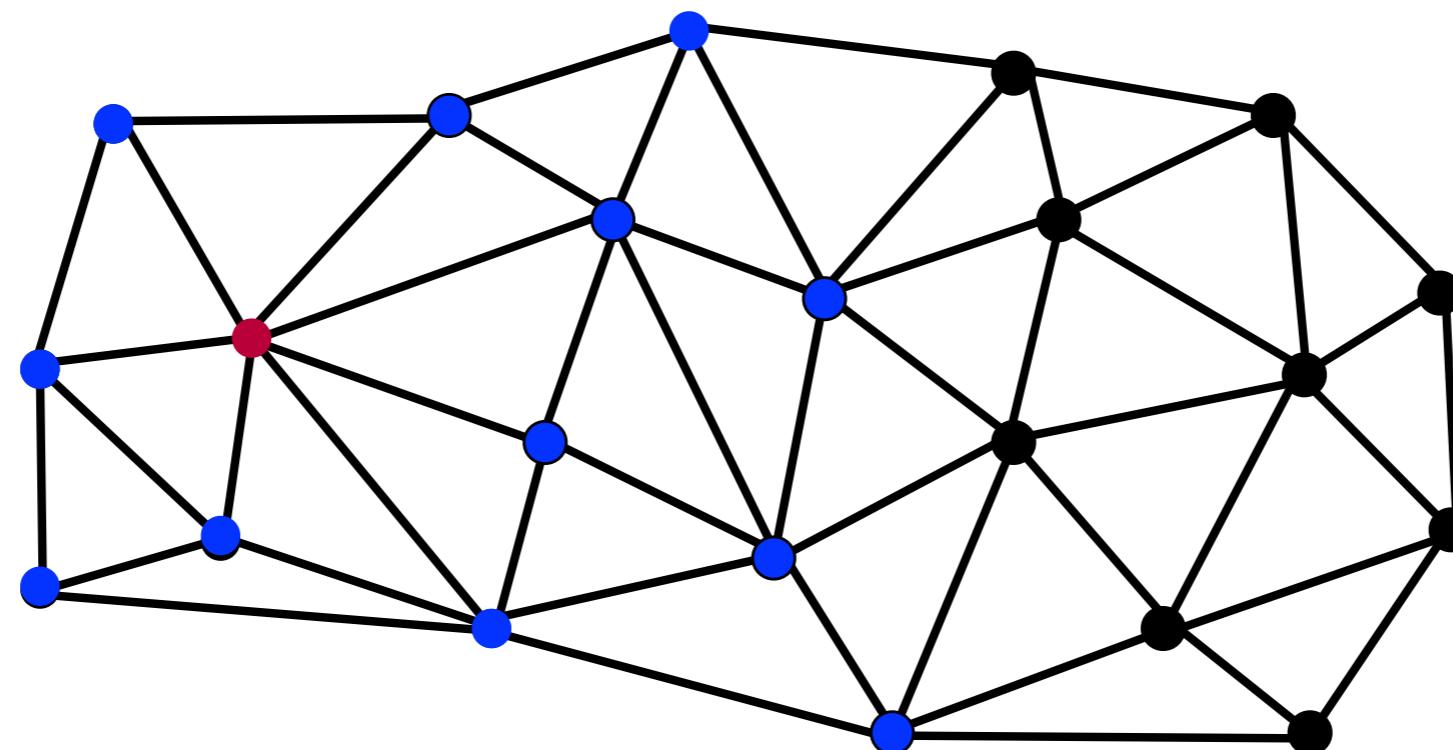
Outline



- Introduction
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- *Online goals:*
 - minimize computing cores
 - minimize communication overhead
 - minimize computational complexity
 - reuse existing codes (data structures, etc.)
- *Observation:* GNAT requires only a few rows of residual/Jacobian
- *Approach:* generate a minimal computational mesh





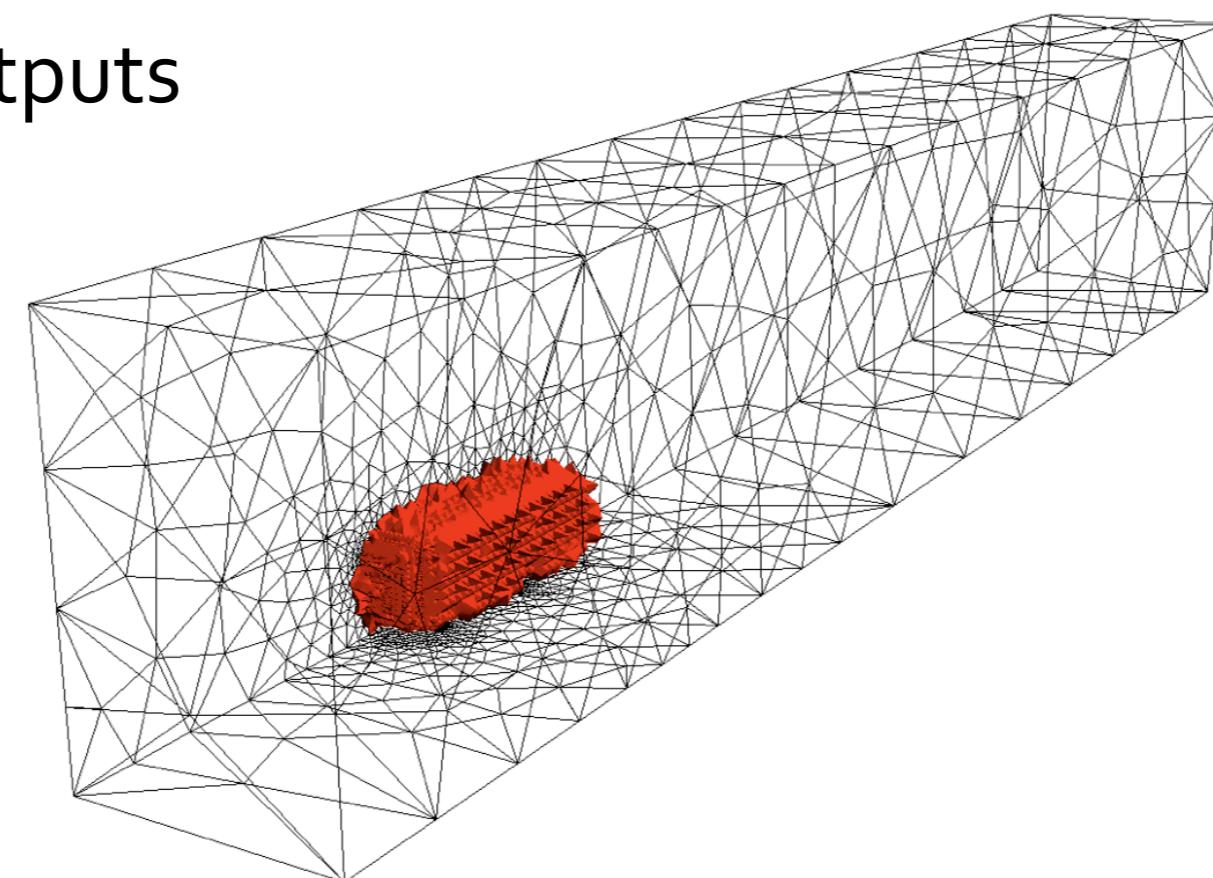
1. **Sample nodes:** compute $R^{(k)}$ and $J^{(k)} \Phi_y$ for all conserved variables
 - see paper for sample node selection algorithm
2. **Neighbor nodes:** compute state
3. Volumes
4. Boundary edges and faces
5. Other (e.g. node distances from the wall)

- *Observations:*

- outputs are often defined locally in space (e.g. lift/drag)
- outputs may not be computable on sample mesh

- *Post-processing step:*

1. read POD coefficients y_r computed by GNAT
2. assemble solution on minimal **post-processing mesh**
3. compute outputs

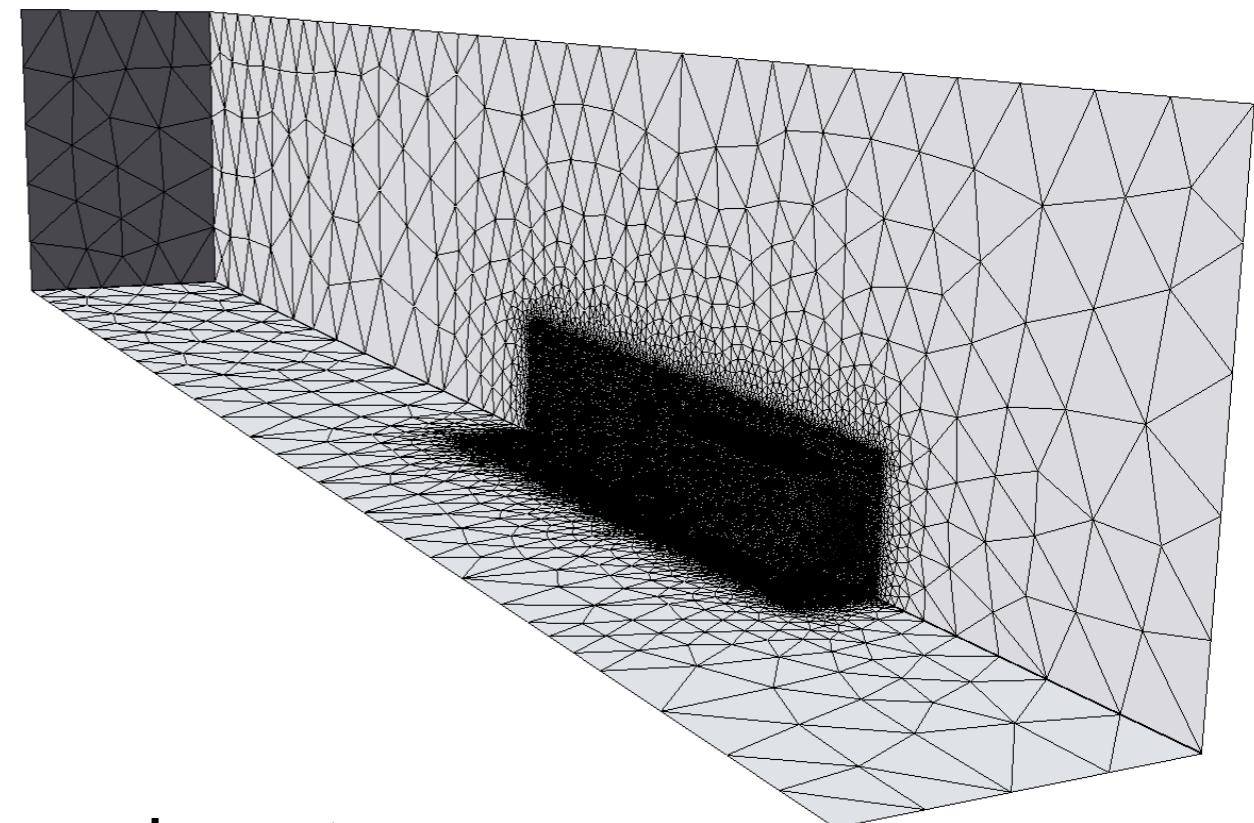
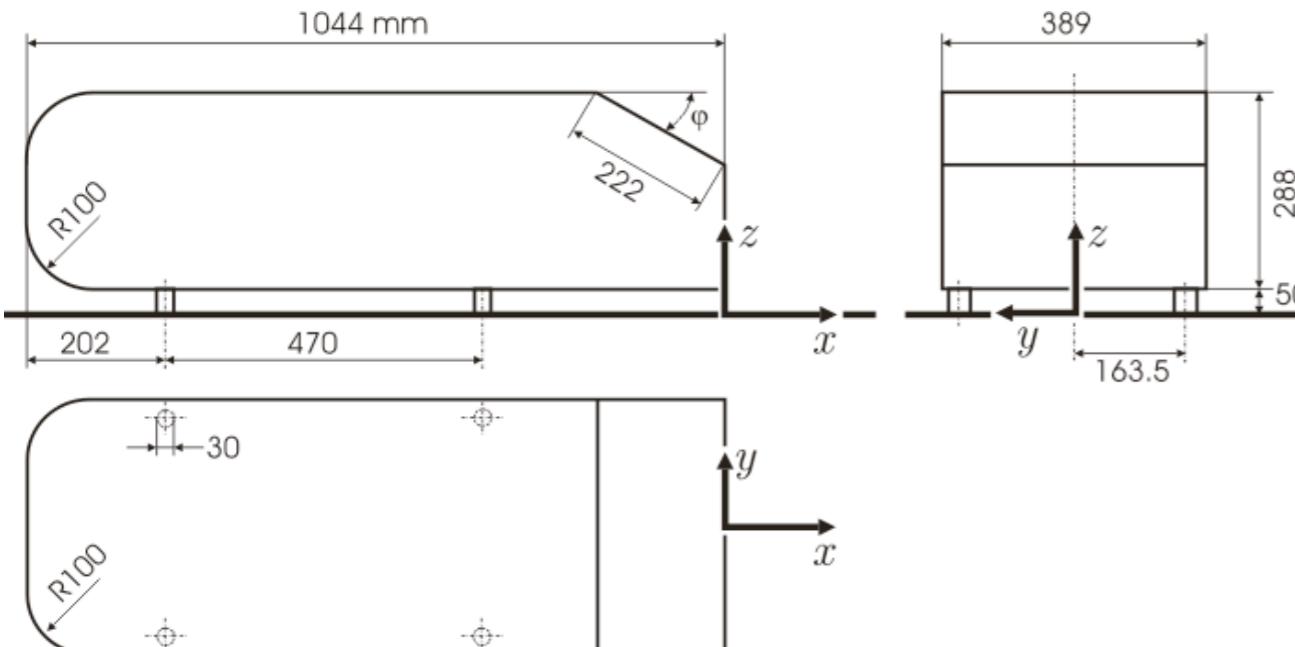


Outline

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- Examples
 - Ahmed body problem
 - Burgers' equation (predictive)



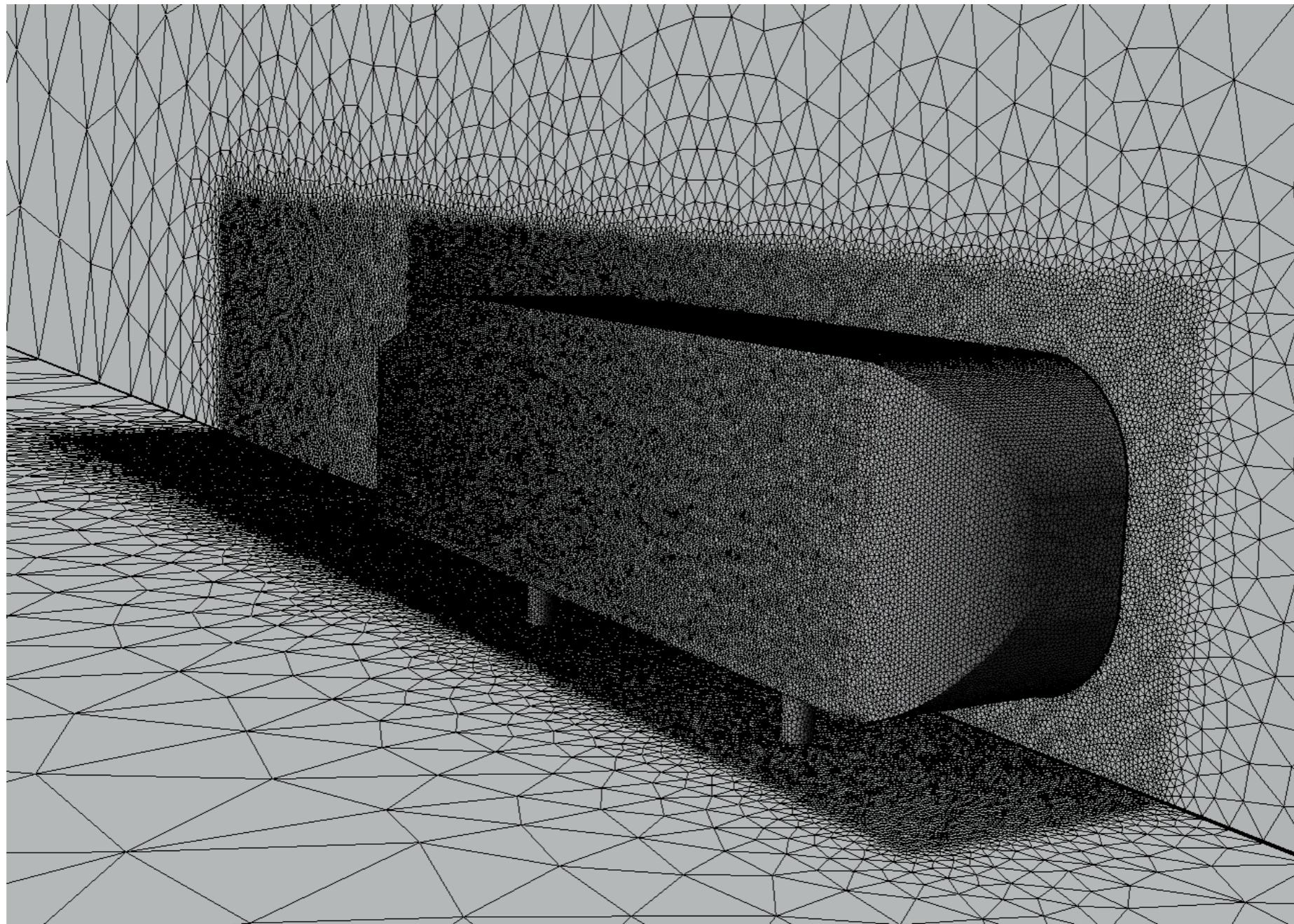
Benchmark: Ahmed body



- Compressible Navier-Stokes (finite volume)

- $\text{Re} = 4.48 \times 10^6$
- $M_\infty = 0.175$ (134 mph)
- 3-point BDF time integration
- CFL=2000
- Steady-state initial condition
- DES turbulence model
- Reichardt's wall law
- V₄ dissipation scheme
- 2nd order flux reconstruction

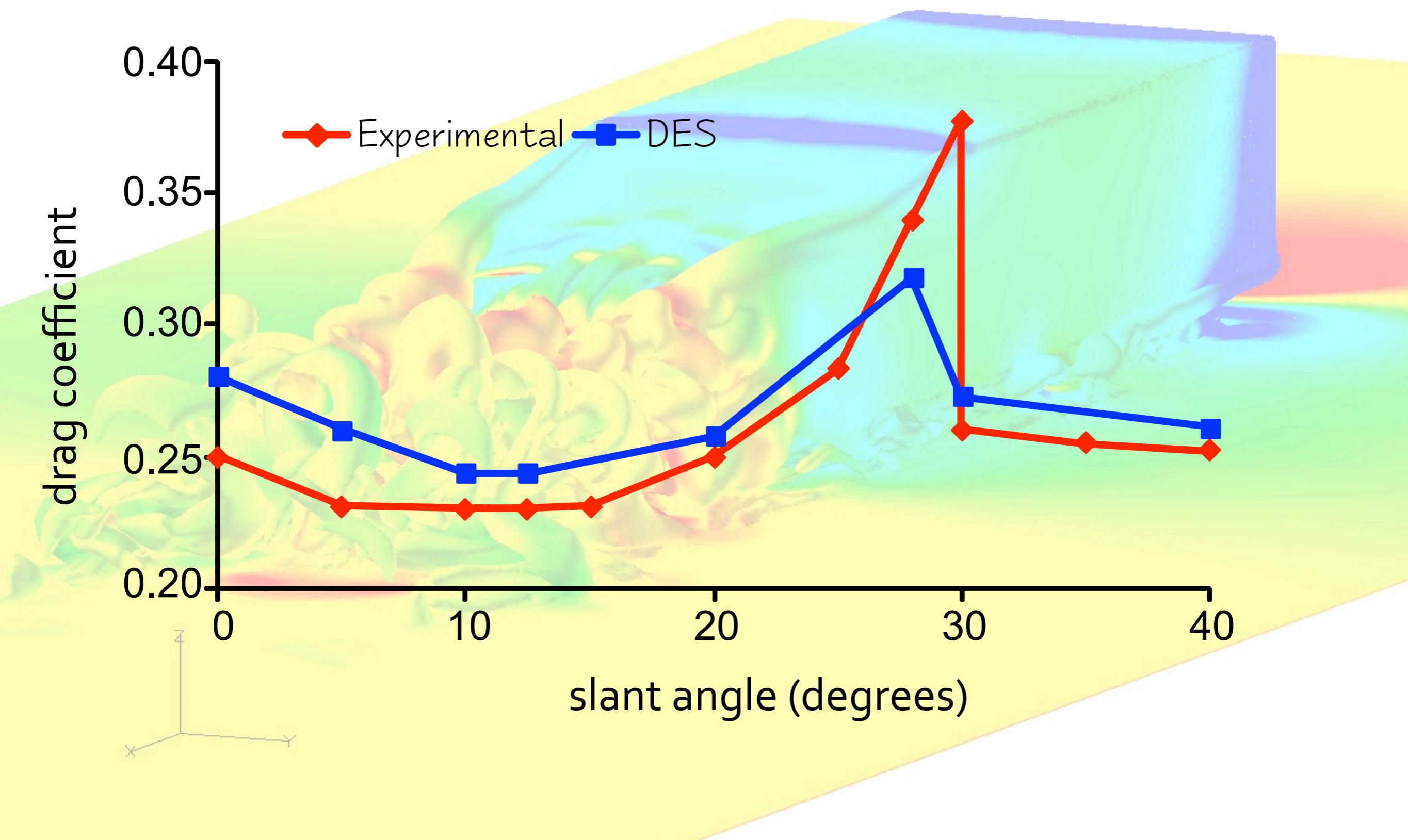
V_∞



- 2.89×10^6 nodes, 1.70×10^7 tetrahedral volumes
- 1.73×10^7 degrees of freedom (dof)

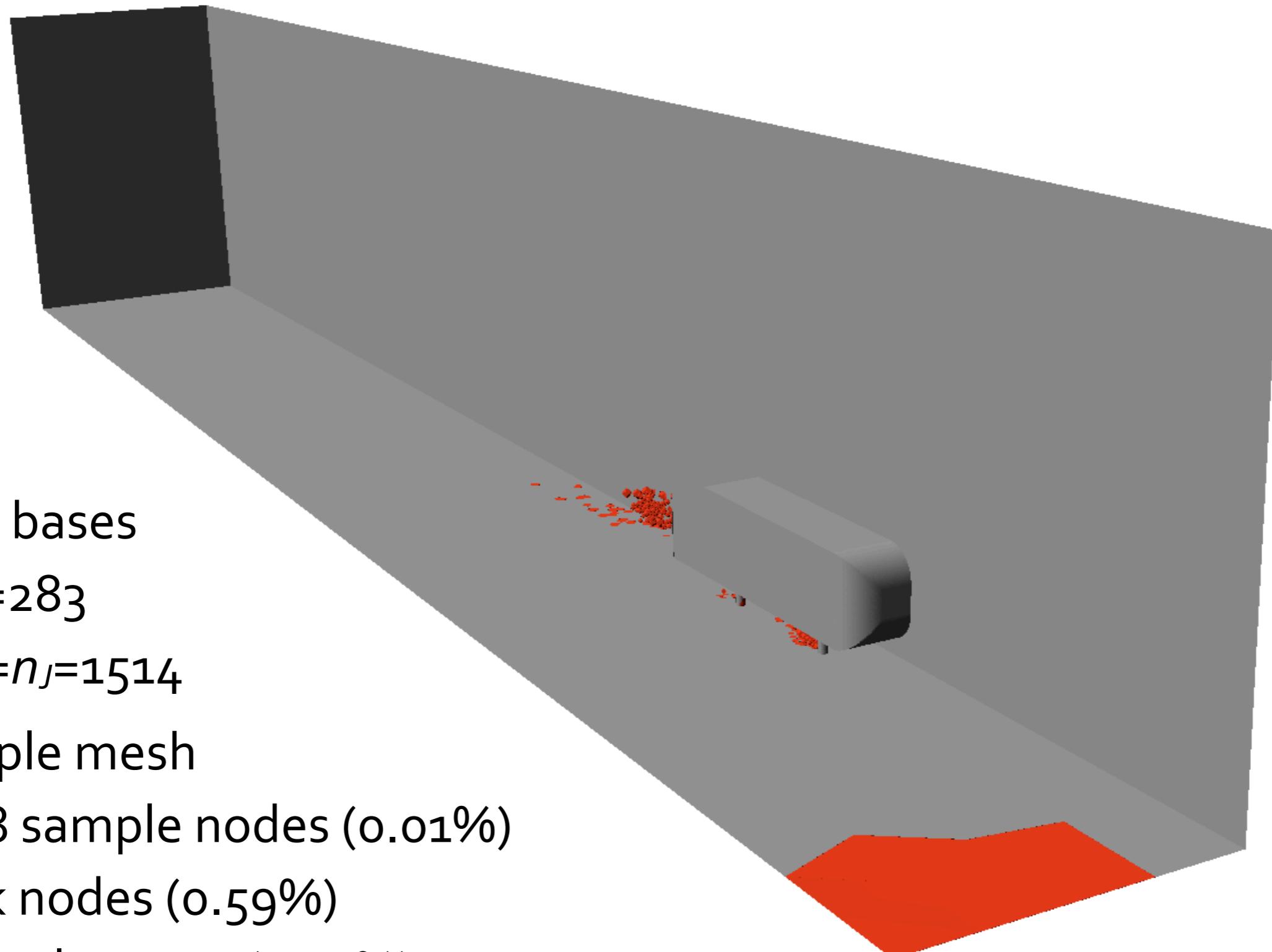


Validation



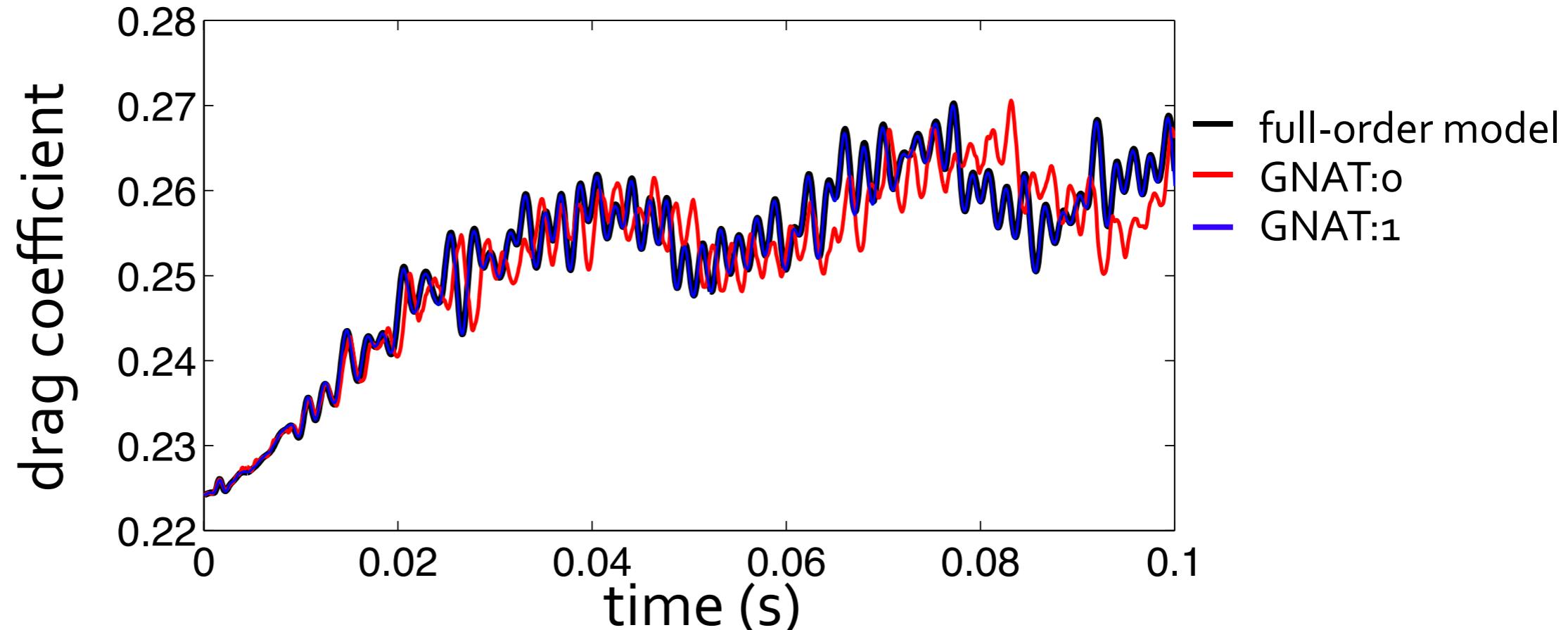


GNAT model





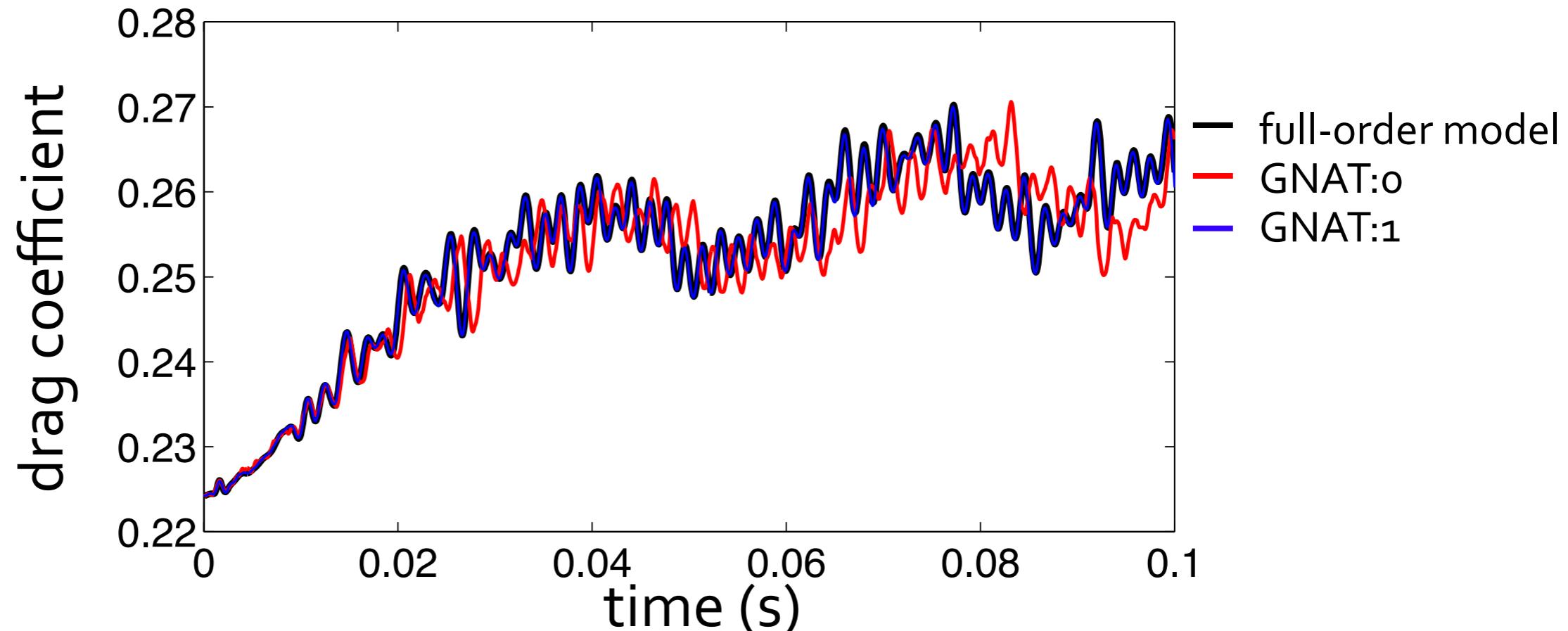
GNAT results



model	consistency conditions	relative error	# cores	time, hours	speedup in cpu resources
FOM	-	-	512	13.3	-
GNAT	0	7.43%	4	7.37	231
GNAT	1	0.68%	4	3.88	438



GNAT results

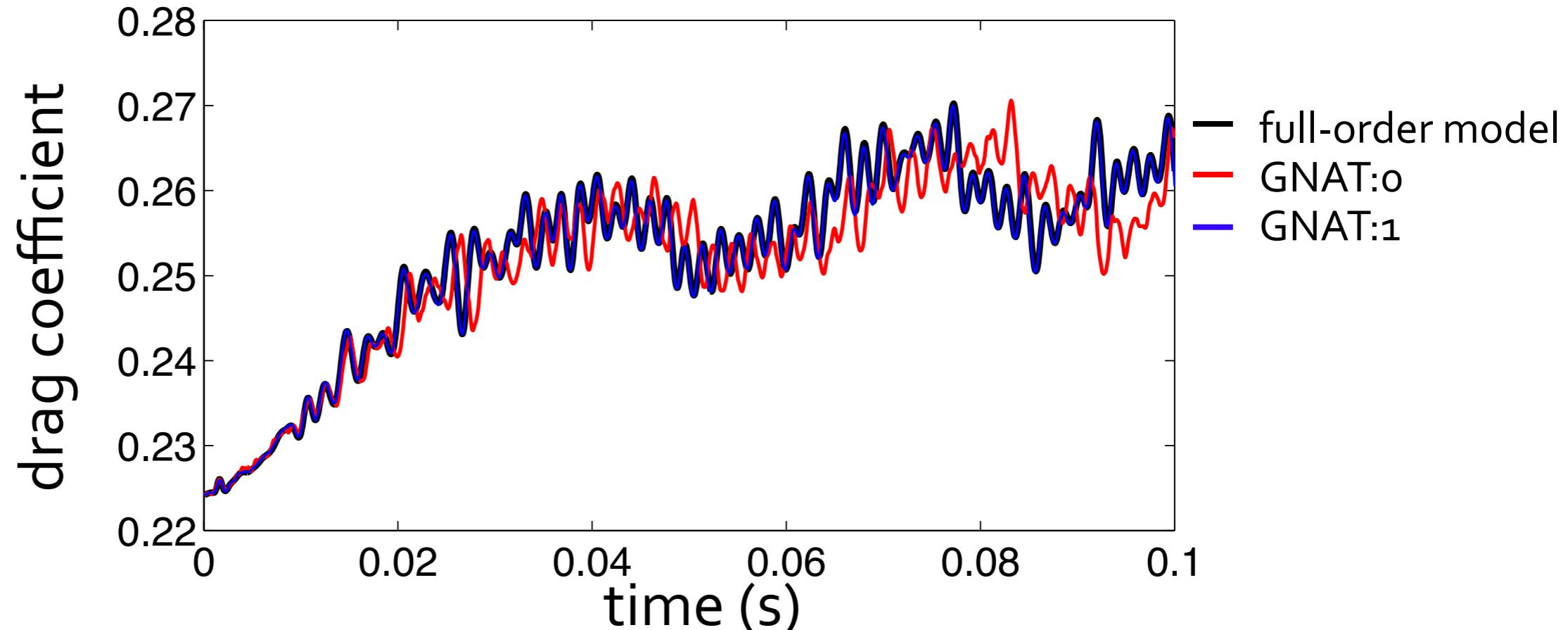


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- **consistency** improves performance



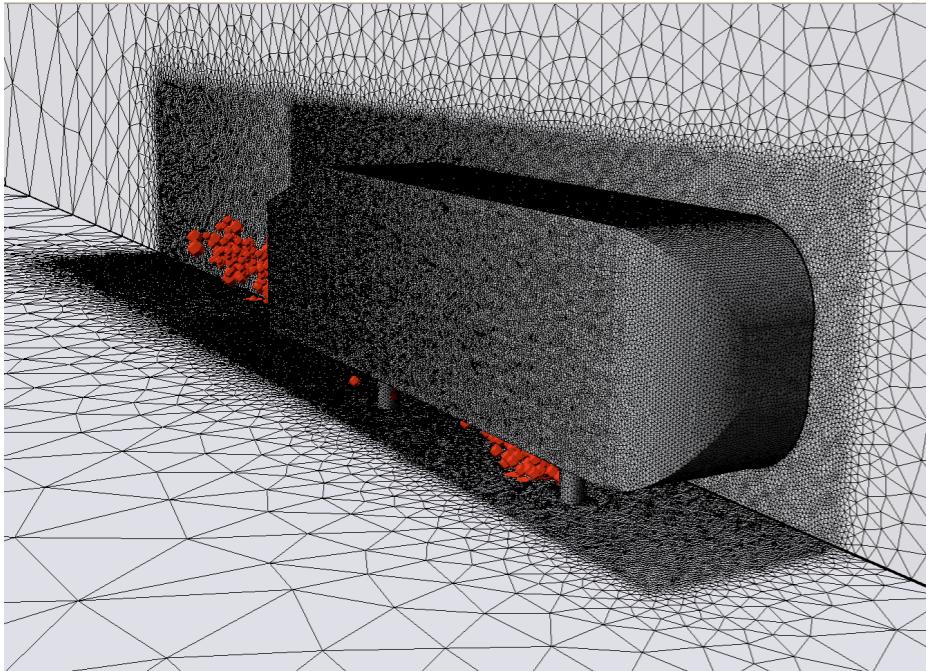
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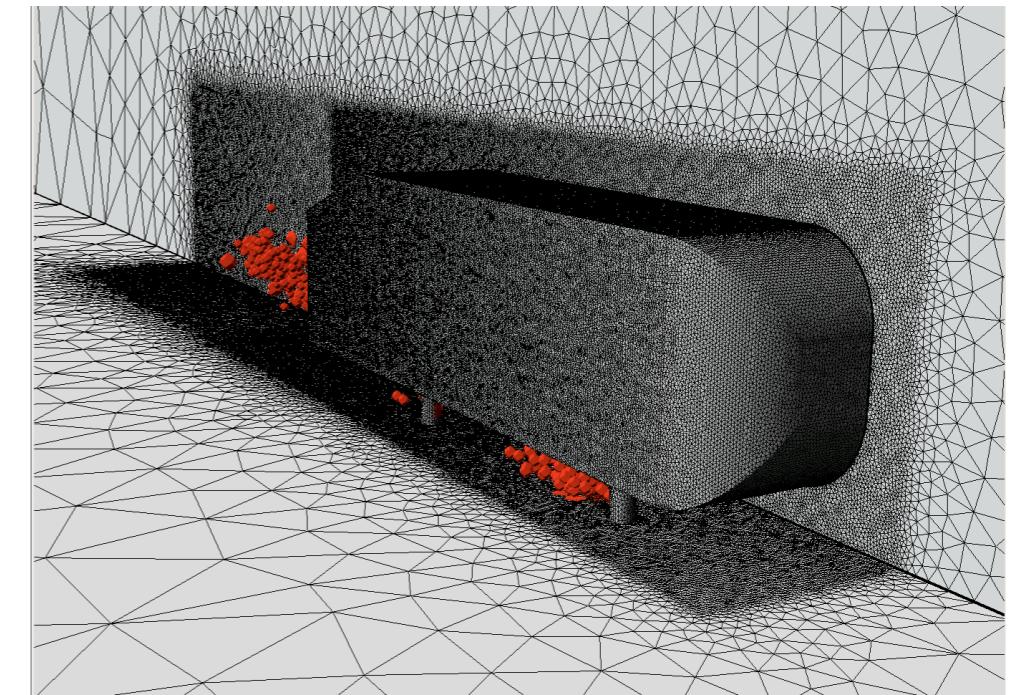
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- consistency improves performance
- sample mesh reduces # cores

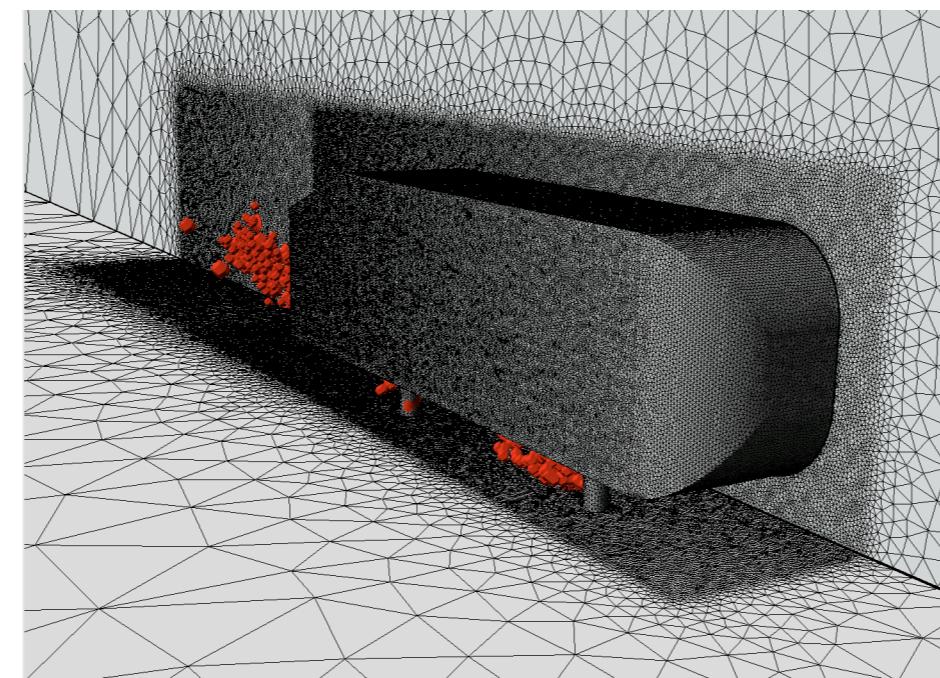
Effect of # sample nodes



253 sample nodes



378 sample nodes



505 sample nodes



Effect of # sample nodes



$$n_y = 283; n_R = n_J = 1514$$

# sample nodes	average # Newton iterations per time step	relative error	speedup in cpu resources
253	4.4	0.79%	452
378	2.8	0.68%	438
505	2.3	0.75%	403

- Adding sample nodes
 - + improves convergence



Effect of # sample nodes



$$n_y = 283; n_R = n_J = 1514$$

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- Adding sample nodes
 - + improves convergence
 - more expensive per iteration



Effect of # sample nodes

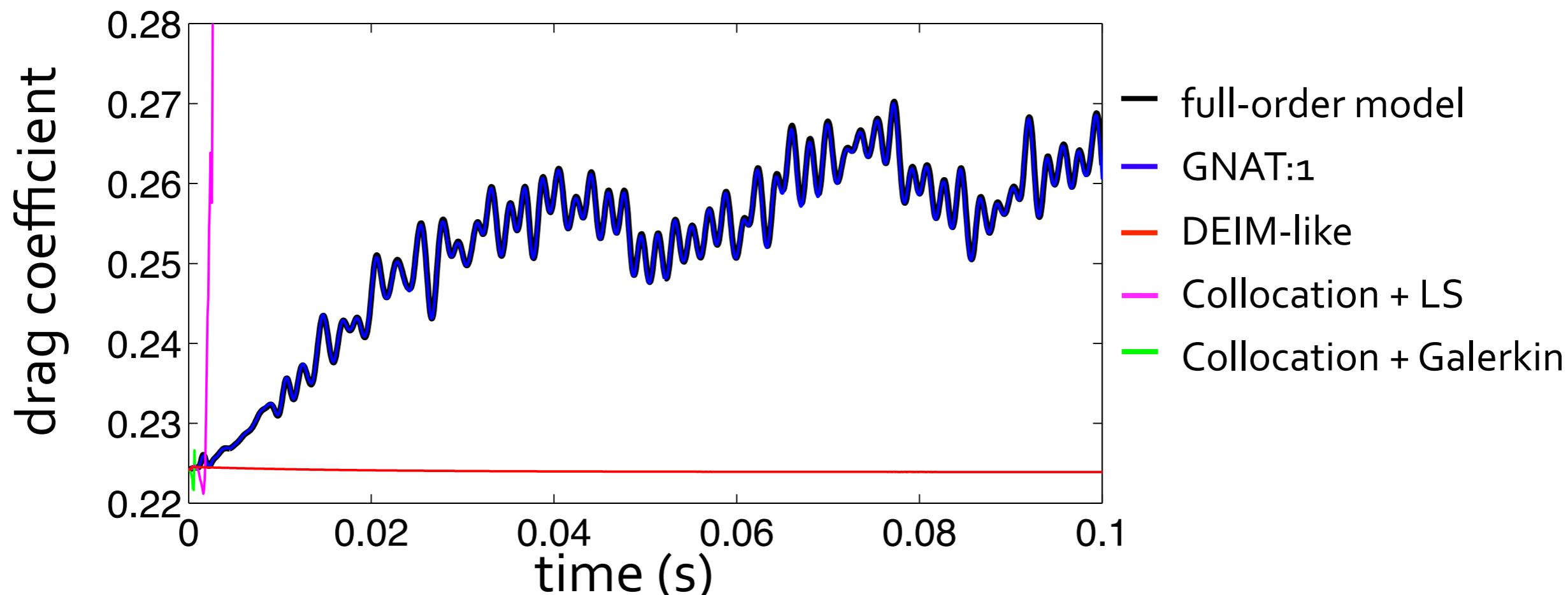


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- Adding sample nodes
 - + improves convergence
 - more expensive per iteration
- **Interpolation** (253 sample nodes) not most effective approach

- Fixed POD basis and sample mesh (378 sample nodes)



- Discrete empirical interpolation (DEIM)-like method fails
 - **Consistency** important, **interpolation** ineffective
- Collocation fails
 - **ignores** unsampled equations



Outline



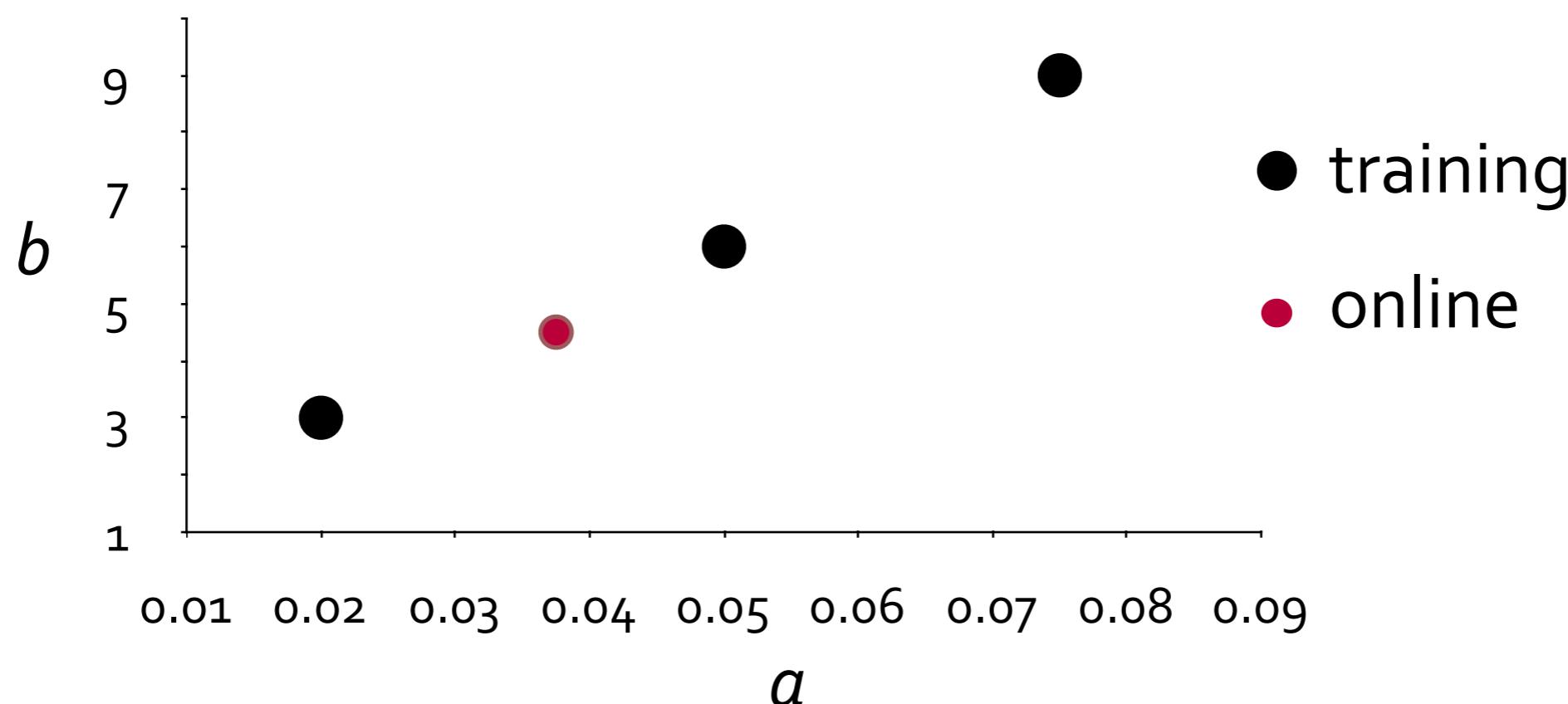
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- Burgers' equation

$$\frac{\partial U(x; t)}{\partial t} + 0.5 \frac{\partial U^2}{\partial x} = 0.02 \exp(ax)$$

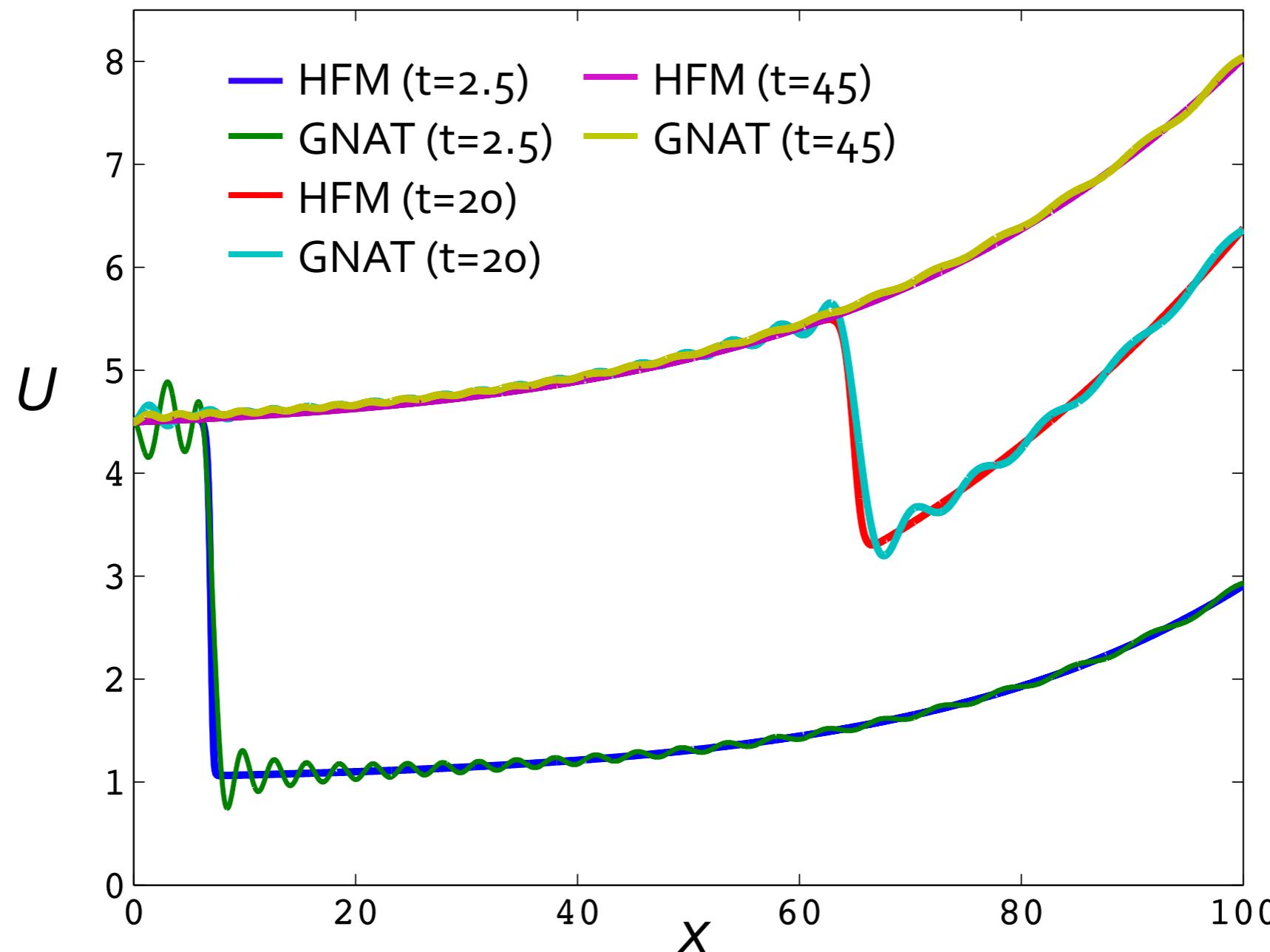
$$U(x, 0) = 1, \quad \forall x \in [0, 100], \quad U(0, t) = b, \quad t > 0$$

- Finite volume discretization (Godunov)
- Full-order model: 4001 dof



• GNAT(2)

- $n_y=50, n_R=160, n_J=70, 160$ sample nodes



+ relative state error = 1.43% + speedup in wall time = 9.4

Summary

- GNAT method
 - Fully discrete framework
 - Consistent & optimal approximations
- Sample mesh concept enables many fewer cores
- Ahmed body example
 - speedups over 350, error less than 1%
 - other complexity reduction methods failed
- Burgers' example established predictive capability



Questions?

Thanks to:

- Collaborators
 - Wade Spurlock
 - Phil Avery
- Funding sources
 - Toyota Motor Sports Division
 - Army High Performance Computing Research Center
 - NSF Graduate Fellowship
 - NDSEG Fellowship

